ZERO SUBSPACES OF POLYNOMIALS ON $\ell_1(\Gamma)$

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ABSTRACT. We provide two examples of complex homogeneous quadratic polynomials P on Banach spaces of the form $\ell_1(\Gamma)$. The first polynomial P has both separable and nonseparable maximal zero subspaces. The second polynomial P has the property that while the index-set Γ is not countable, all zero subspaces of P are separable.

1. INTRODUCTION

Let X be a Banach space. A multilinear form on X of degree n is a multilinear function $A : X^n \longrightarrow Z$ which is moreover continuous. Continuity is equivalent to the existence of a constant K such that $\|A(x_1, \ldots, x_n)\| \leq K\|x_1\| \cdots \|x_n\|$. The minimal such K is the norm of A. The multilinear form A is said to be symmetric if $A(x_1, \ldots, x_n) = A(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$ for every permutation σ. A homogenous polynomial of degree n on X is a function $P : X \longrightarrow Z$ of the form $P(x) = A(x, \ldots, x)$ for A a symmetric multilinear form on X. This multilinear form A is uniquely determined by P. In general, a polynomial is a sum of homogenuous polynomials and constants (constant functions are considered as homogenous polynomials of degree 0). A homogeneous polynomial of degree two is called a quadratic functional.

A zero subspace of a polynomial P is a subspace $Y \subset X$ such that $P(y) = 0$ for every $y \in Y$. For homogeneous polynomial, this is equivalent to say that $A(y_1, \ldots, y_n) = 0$ for all $y_1, \ldots, y_n \in Y$, where A is the symmetric multilinear form associated to P. The study of zero subspaces of polynomials depends heavily on whether the field of scalars is the field of real numbers or the field complex numbers[1]. For example, Aron and Hajek [2] has shown that every Banach space X with weak-star separable dual admits a homogeneous polynomial $P : X \longrightarrow \mathbb{R}$ of degree 3 with no infinite-dimensional null space. It follows that for example the Banach spaces ℓ_∞ and $\ell_1(\Gamma)$ for Γ an index set of cardinality continuum admit homogeneous polynomials of degree 3 with no infinite-dimensional null spaces. On the other hand, for complex Banach spaces (on which we will concentrate in this paper), Plichko and Zagorodnyuk [7] have proved the following result.

Theorem 1. *Let X be an infinite-dimensional complex Banach space and let $P : X \longrightarrow \mathbb{C}$ be a polynomial with $P(0) = 0$. Then, there is an infinite-dimensional*

[1]For more information about zero subspaces of polynomials on infinite dimensional Banach spaces over $\mathbb{R}$, we refer to [1] and the recent work of Aron and Hájek [2].

zero subspace $Y \subset X$ for P.

As indicated above, in this paper we are mainly concerned with polynomials with values in the field of complex numbers, so from now on all Banach spaces that we will be working with will be assumed to be over the field of complex numbers. Note that Theorem 1 suggests several natural questions concerning zero subspaces of polynomials when the Banach space X is assumed to be nonseparable, and not just infinite-dimensional. Most prominent of these are the following two problems.[2]

Problem 2. *Let X be a nonseparable Banach space and $P : X \longrightarrow \mathbb{C}$ a polynomial with $P(0) = 0$. Is there always a nonseparable zero subspace for P?*

Problem 3. *Let X be a nonseparable Banach space and $P : X \longrightarrow \mathbb{C}$ a polynomial with $P(0) = 0$. May P have a separable maximal zero subspace? Do all maximal zero subspaces of P have the same density?*

In this note, we shall show that the answer to both questions is in general negative by providing the respective counterexamples as quadratic functionals on the complex version of the Banach space $\ell_1(\omega_1)$. It should be noted that the first problem has already some history behind. Banakh, Plichko and Zagorodnyuk [6] noticed that if X^* is weak* nonseparable, then every quadratic functional on X has a nonseparable zero subspace, and this is the case for instance of $\ell_1(\mathfrak{c}^+)$ and of any nonseparable reflexive space. Using this fact, Fernández-Unzueta [5] proved that also every quadratic functional on ℓ_∞ has a nonseparable zero subspace. Generalizing that result of [6], we notice that if the weak*-density of X^* is greater than $\beth_{m-2}$, then every homogeneous polynomial $P : X \longrightarrow \mathbb{C}$ of degree m has a nonseparable zero subspace. In this regard, we may pose the following problem:

Problem 4. *Let $m > 1$. Is there a homogeneous polynomial $P : \ell_1(\beth_{m-1}) \longrightarrow \mathbb{C}$ of degree m such that all its zero subspaces are separable?*

They explicitly ask in [6] whether there is a quadratic functional $P : \ell_1(\omega_1) \longrightarrow \mathbb{C}$ with all zero subspaces separable and this is precisely what we construct. The construction of the counterexample to problem 3 is rather standard and uses large almost disjoint families of sets. On the other hand, the counterexample for question 2 requires the existence certain partitions $f : [\Gamma]^2 \to \{0, 1\}$ with special properties[3]. The proof of the existence of such partitions relies on the machinery of minimal walks on countable ordinals from [8]. These partitions can also be produced in a forcing extensions of the universe of sets, showing, for example, the consistency of the existence of a quadratic functional with no nonseparable zero subspaces exists over spaces of the form $X = \ell_1(\Gamma)$ for Γ an index set of cardinality continuum. This leads us to the following natural question to be answered if possible without appealing to special axioms of set theory.

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[2] Reiterated, for example, in the recent lecture by R. Aron at the Caceres Conference on Banach Space Theory of September 2006.

[3]It should be noted however that use of partitions of this kind for constructing polynomials appears first in the paper of Hajek and Aron [2].

Problem 5. *Is there a polynomial $P : \ell_1(\mathfrak{c}) \longrightarrow \mathbb{C}$ with $P(0) = 0$ with no nonseparable zero subspace?*

Using another related special property of partitions arising again from the work of the second author [8], we produce some vector-valued polynomials with even stronger properties. For example, we show that for every separable Banach space V there exists a quadratic functional $P : \ell_1(\omega_1) \longrightarrow V$ such that the closure of the image under P of any nonseparable subspace of $\ell_1(\omega_1)$ has nonempty interior. Asking for this kind of extension of our basic example is motivated by the work of Baumgartener and Spinas [3] which also uses the partitions of [8] to produce complex bilinear forms in the context of vector spaces over a countable field rather than in the context of Banach spaces over the field $\mathbb{C}$

2. GENERAL REMARKS ABOUT POLYNOMIALS

The examples that we construct are defined on spaces $\ell_1(\Gamma)$. Over this space, all polynomials can be explicitly described. For instance, the general form of a quadratic functional $P : \ell_1(\Gamma) \longrightarrow \mathbb{C}$ is:

$$P(x) = \sum_{\alpha,\beta \in \Gamma} \lambda_{\alpha\beta} x_\alpha x_\beta, \quad x = (x_\gamma)_{\gamma \in \Gamma} \in \ell_1(\Gamma),$$

where $(\lambda_{\alpha\beta})_{\alpha,\beta \in \Gamma}$ is a bounded family of complex scalars. Indeed in all our examples the coefficients $\lambda_{\alpha\beta}$ are either 0 or 1, they functionals of the form:

$$P(x) = \sum_{\{\alpha,\beta\} \in G} x_\alpha x_\beta,$$

where G a certain set of couples of elements of Γ. On the other hand, we state the following elementary basic fact about polynomials that we shall explicitly use at some point:

Proposition 6. *Let $P : X \longrightarrow Y$ be a homogeneous polynomial of degree n and norm K. Then $\|P(x) - P(y)\| \leq nKM^{n-1}\|x\|\|y\|$ for every $x, y \in X$.*

3. POLYNOMIALS WITH BOTH SEPARABLE AND NONSEPARABLE MAXIMAL ZERO SUBSPACES

Let Ω be a set and $\mathcal{A}$ be an almost disjoint family of subsets of Ω (that is, $|A \cap A'| < |\Omega|$ whenever $A, A' \in \mathcal{A}$ are different), and let $\mathcal{B} = \Omega \cup \mathcal{A}$. We consider the following quadratic functional $P : \ell_1(\mathcal{B}) \longrightarrow \mathbb{C}$ given by

$$P(x) = \sum \{x_n x_A : n \in \Omega, A \in \mathcal{A}, n \in A\}.$$

Theorem 7. *The space $X = \ell_1(\Omega) \subset \ell_1(\mathcal{B})$ is maximal zero subspace for the polynomial P.*

Proof: The only point which requires explanation is that X is indeed maximal. So assume by contradiction that there is a vector y out of X such that $Y = span(X \cup \{y\})$ is a zero subspace for P. Without loss of generality, we suppose that y is supported in $\mathcal{A}$. Pick $A \in \mathcal{A}$ such that $|y_A| = max\{|y_B| : B \in \mathcal{A}\}$ and $\mathcal{F} \subset \mathcal{A}$ a

finite subset of $\mathcal{A}$ such that $\sum_{B \in \mathcal{A} \setminus \mathcal{F}} |y_B| < \frac{1}{9}|y_A|$. Now, because $\mathcal{A}$ is an almost disjoint family of subsets of Ω, it is possible to find $n \in \Omega$ such that $n \in A$ but $n \notin B$ whenever $B \in \mathcal{F} \setminus \{A\}$. Consider the element $y + 1_n \in Y$. We claim that $P(y + 1_n) \neq 0$ getting thus a contradiction.

$$P(y + 1_n) = \sum_{n \in B} y_B = y_A + \sum_{n \in B, B \in \mathcal{A} \setminus \mathcal{F}} y_B$$

The second term of the sum has modulus less than $\frac{1}{9}$ the modulus of the first term. So $P(y + 1_n) \neq 0$. □

We are interested in the case when $|\mathcal{A}| > |\Omega|$. The subspace $\ell_1(\mathcal{A})$ is a zero subspace for P. It may not be maximal but this does not matter because, by a Zorn's Lemma argument, it is contained in some maximal zero subspace. This fact together with Theorem 7 shows that P has maximal zero subspaces of both densities $|\Omega|$ and $|\mathcal{A}|$.

There are two standard constructions of big almost disjoin families. One is by induction, and it shows that for every cardinal κ we can find an almost disjoint family of cardinality κ^+ on a set of cardinality κ. The other one is by considering the branches of the tree $\kappa^{<\omega}$, and this indicates that for every cardinal κ we can find an almost disjoint family of cardinality κ^ω (One construction or the other provides a better result depending on whether $\kappa^\omega = \kappa$, $\kappa^\omega = \kappa^+$ or $\kappa^\omega > \kappa^+$). Hence,

Corollary 8. *Let κ be an infinite cardinal and $\tau = \max(\kappa^+, \kappa^\omega)$. There exists a quadratic functional on $\ell_1(\tau)$ with a maximal zero subspace of density κ and another maximal zero subspace of density τ.*

Corollary 9. *There exists a quadratic functional on $\ell_1(\mathfrak{c})$ with a separable maximal zero subspace and a maximal zero subspace of density $\mathfrak{c}$.*

4. Polynomials for which all zero subspaces are separable

We denote by $[A]^2$ the set of all unordered pairs of elements of A,

$$[A]^2 = \{t \subset A : |t| = 2\}.$$

We consider an ordinal α to be equal to the set of all ordinals less than α, so

$$\omega_1 = \{\alpha : \alpha < \omega_1\}$$

is the set of countable ordinals, and also for a non negative integer $n \in \mathbb{N}$,

$$n = \{0, 1, \ldots, n-1\}.$$

We introduce some notations for subsets of a well ordered set Γ. If $a \subset \Gamma$ is a set of cardinality n, and $k < n$ we denote by $a(k)$ the $k + 1$-th element of a according to the well order of Γ, so that

$$a = \{a(0), \ldots, a(n-1)\}.$$

Moreover, for $a, b \subset \Gamma$, we write $a < b$ if $\alpha < \beta$ for every $\alpha \in a$ and every $\beta \in b$.

We recall also that a Δ-system with root a is a family of sets such that the intersection of every two different elements of the family equals a. The well known Δ-system lemma asserts that every uncountable family of finite sets has an uncountable subfamily which forms a Δ-system (see, e.g., [4]).

Definition 10. A function $f : [\Gamma]^2 \longrightarrow 2$ is said to be a *partition of the first kind* if for every uncountable family A of disjoint subsets of Γ of some fixed finite cardinality n, and for every $k \in n$ there exist $a, b, a', b' \in A$ such that $f(a(k), b(k)) = 1$, $f(a'(k), b'(k)) = 0$ and $f(a(i), b(j)) = f(a'(i), b'(j))$ whenever $(i, j) \neq (k, k)$. Notice that, passing to a further uncountable subfamily A, we can choose such $a < b$ such that, in addition, $f(a(i), a(j)) = f(a'(i), a'(j)) = f(b(i), b(j)) = f(b'(i), b'(j))$ for all $\{i, j\} \in [n]^2$.

A natural projection of the square-bracket operation of [8] (see also [9]) gives the following.

Theorem 11. *For $\Gamma = \omega_1$ there is a partition $f : [\Gamma]^2 \longrightarrow 2$ of the first kind.*

Remark 12. Note that if some index set Γ admits a partition $f : [\Gamma]^2 \longrightarrow 2$ of the first kind then its cardinality is not more than the continuum. So one would naturally like to know if some index set Γ of cardinality continuum admits such a partition. While at the moment we do not know the answer to this problem, we do know that the natural forcing of finite approximations to $f : [\Gamma]^2 \longrightarrow 2$ leads to a generic extension of the universe of sets which admits a partition $f : [\Gamma]^2 \longrightarrow 2$ of the first kind on some index set Γ of cardinality continuum that is considerably larger than $\aleph_1$.

To a given a partition $f : [\Gamma]^2 \longrightarrow 2$, we associate the polynomial on $\ell_1(\Gamma)$ given by

$$P_f(x) = \sum_{\{i,j\} \in f^{-1}(1)} x_i x_j.$$

Theorem 13. *If $f : [\Gamma]^2 \longrightarrow 2$ is a partition of the first king and if Y is a subspace of $\ell_1(\Gamma)$ with $Y \subset P_f^{-1}(0)$, then Y is separable.*

Proof: We simplify the notation by writing P in place of P_f and we assume on the contrary that we have $Y \subset P^{-1}(0)$ a nonseparable space. We can construct recursively a transfinite sequence $(\beta^i : i < \omega_1)$ of different elements of Γ and vectors $(y^i : i < \omega_1)$ with $y^i \in Y$, $\|y^i\| = 1$ and $y^i_{\beta^i} \neq 0$. By passing to a cofinal subsequence, we can assume that there exists $\varepsilon \in (0, 1)$ such that $|y^i_{\beta^i}| > \varepsilon$ for every $i < \omega_1$. This ε will be fixed from now on in the proof. We can also suppose that the sequence (β^i) is increasing. Since a polynomial is uniformly continuous on any bounded set (Proposition 6), we pick a δ such that $|P(x) - P(y)| < \frac{\varepsilon^2}{8}$ whenever $\|x - y\| < \delta$ and $\|x\|, \|y\| \leq 3$.

For every $i < \omega_1$ we fix a finite set $a_i \subset supp(y^i)$ such that $\beta^i \in a_i$ and

$$\sum_{u \notin a_i} |y^i_u| < \delta/3.$$

In other words, if we call x^i to the vector which coincides with y^i on the coordinates of a_i and is supported on a_i, we have that $\|x^i - y^i\| < \delta/3$. By virtue of the Δ-system lemma, passing to a further cofinal subsequence we can suppose that the sets $\{a_i : i < \omega_1\}$ form a Δ-system with root a. We call $b_i = a_i \setminus a$. Again, by passing to a cofinal subsequence, we may suppose that:

- There exists $n < \omega$ such that $|b_i| = n$ for every $i < \omega_1$.

- There exists $k \in n$ such $\beta^i = b_i(k)$ for every $i < \omega_1$.

- There exists a function $e : a \times n \longrightarrow 2$ such that $f\{\alpha, b_i(r)\} = e(\alpha, r)$ for every $\alpha \in a$ and $r \in n$.

- There exists a function $d : [n]^2 \longrightarrow 2$ such that $f\{b_i(r), b_i(s)\} = d\{r, s\}$ for every $\{r, s\} \in [n]^2$ and $i < \omega_1$.

For every possible function $q : n \times n \longrightarrow 2$ we consider the continuous function $\phi_q : \mathbb{C}^{a \cup n} \times \mathbb{C}^{a \cup n} \longrightarrow \mathbb{C}$ given as follows:

$$
\begin{aligned}
\phi_q(z, z') \;\; = \;\; & \sum \{(z_\alpha + z'_\alpha)(z_\beta + z'_\beta) : \{\alpha, \beta\} \in [a]^2, f\{\alpha, \beta\} = 1\} + \\
& \sum \{(z_\alpha + z'_\alpha)(z_r + z'_r) : \alpha \in a, r \in n, e(\alpha, r) = 1\} + \\
& \sum \{z_r z_s + z'_r z'_s : \{r, s\} \in [n]^2, d\{r, s\} = 1\} + \\
& \sum \{z_r z'_s : (r, s) \in n^2, q(r, s) = 1\}.
\end{aligned}
$$

That is, $\phi_q(z, z')$ computes the evaluation of the polynomial P on a hypothetical vector of the form $x^i + x^{i'}$ provided that the restriction of x^i to a_i is given by z, the restriction of $x^{i'}$ to $a_{i'}$ is given by z', and the value of $f(b_i(r), b_{i'}(s))$ is given by q. Let $\mathcal{B}$ be a countable base for the topology of the space $\mathbb{C}^{a \cup n}$. For every $i < \omega_1$ we can choose $U_i \in \mathcal{B}$ to be a neighborhood of the vector

$$
y^i_* = (y^i_{a(0)}, \ldots, y^i_{a(|a|-1)}, y^i_{b_i(0)}, \ldots, y^i_{b_i(n-1)})
$$

such that $|\phi_q(z) - \phi_q(w)| < \frac{\varepsilon^2}{8}$ for every $z, w \in U_i \times U_i$ and every possible q. Since $\mathcal{B}$ is countable, we can make a further assumption:

- There exists $U \in \mathcal{B}$ such that $U_i = U$ for every $i < \omega_1$.

Now, we use the fundamental property of f to find ordinals $i < j$ and $i' < j'$ such that $f\{b_i(k), b_j(k)\} = 1$, $f\{b_{i'}(k), b_{j'}(k)\} = 0$ and in all other cases, $f\{b_i(r), b_j(s)\} = f\{b_{i'}(r), b_{j'}(s)\}$.

We claim that $P(y^i + y^j) \neq P(y^{i'} + y^{j'})$, which finishes the proof since these two vectors belong to Y, and this way they cannot be both zero.

Notice that $|P(x^{i'} + x^{j'}) - P(y^{i'} + y^{j'})| < \frac{\varepsilon^2}{8}$ and $|P(x^i + x^j) - P(y^i + y^j)| < \frac{\varepsilon^2}{8}$ since we know that $\|x^u - y^u\| < \delta/3$ for every u and also $|P(x) - P(y)| < \frac{\varepsilon^2}{8}$ whenever $\|x - y\| < \delta$ and $\|x\|, \|y\| \leq 3$.

Define a function $q : n^2 \longrightarrow 2$ as $q(r, s) = f\{b_{i'}(r), b_{j'}(s)\}$. We can compute
$$
\begin{aligned}
P(x^{i'} + x^{j'}) &= \phi_q(y^{i'}_*, y^{j'}_*), \\
P(x^i + x^j) &= \phi_q(y^i_*, y^j_*) + y^i_{b_i(k)} y^j_{b_j(k)}.
\end{aligned}
$$

On the one hand, $y^i_{b_i(k)} y^j_{b_j(k)} = y^i_{\beta^i} y^j_{\beta^j}$, so $|y^i_{b_i(k)} y^j_{b_j(k)}| > \varepsilon^2$.

On the other hand, since $y^{i'}_*, y^{j'}_*, y^i_*, y^j_* \in U$, we have that
$$
|\phi_q(y^i_*, y^j_*) - \phi_q(y^{i'}_*, y^{j'}_*)| < \frac{\varepsilon^2}{8}.
$$
All these inequalities together yield that $|P(y^{i'} + y^{j'}) - P(y^i + y^j)| > \frac{\varepsilon^2}{2}$. $\quad\square$

5. Polynomial functionals with ranges in separable Banach spaces

We shall denote by
$$
\Delta_n = \{(i, i) : i \in n\}
$$
the diagonal of the cartesian product $n \times n$.[4] Also $B_X(x, r)$ or simply $B(x, r)$ will denote the ball of center x and radius r in a given Banach space X.

Definition 14. A function $f : [\Gamma]^2 \longrightarrow \omega$ is said to be a *partition of the second kind* if for every uncountable family A of finite subsets of Γ all of some fixed cardinality n, we have the following two conclusions

(a) there is an uncountable subfamily B of A and a function $h : n^2 \setminus \Delta_n \longrightarrow \omega$ such that $f(a(i), b(j)) = h(i, j)$ for every $i \neq j$, $i, j < n$ and every $a < b$ in B;

(b) for every function $h : n \longrightarrow \omega$ there exists $a < b$ in A such that $f(a(i), b(i)) = h(i)$.

A natural projection of the square-bracket operation of [8] leads again to the following result (see also [9]).

Theorem 15. *For $\Gamma = \omega_1$ there is a partition $f : [\Gamma]^2 \longrightarrow 2$ of the second kind as well.*

Remark 16. Clearly, every such a set must have a cardinality no bigger than the continuum. Similarly to the case of partitions of the first kind, we know that it is consistent to have partitions of the second king on some index set Γ of cardinality continuum that is bigger than $\aleph_1$ though this cannot be achieved by going to a generic extension of the poset of finite approximations to such a partition.

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[4]Recall our convention, $n = \{0, 1, ..., n - 1\}$.

Fix a separable space V and $\{v_n : n < \omega\}$ a dense subset of the unit ball of V with $v_0 = 0$. Then to every partition $f : [\Gamma]^2 \longrightarrow \omega$ we associate a homogeneous polynomial $P_f : \ell_1(\Gamma) \longrightarrow V$ of degree 2 by the formula

$$P_f(x) = \sum_{\{i,j\} \in [\Gamma]^2} x_i x_j v_{f(i,j)}$$

Theorem 17. *Suppose that $f : [\omega_1]^2 \longrightarrow \omega$ is a partition of the second kind and let $P = P_f : \ell_1(\Gamma) \longrightarrow V$ be the corresponding polynomial. Let Y be a nonseparable subspace of $\ell_1(\omega_1)$. Then $\overline{P(Y)}$ has nonempty interior in V.*

Proof: Let us introduce some language. Given $G, H \subset V$ and $\lambda > 0$, we say that G is λ-dense in H if for every $h \in H$ there exists $g \in G$ such that $\|g - h\| < \lambda$.

We can construct recursively a transfinite sequence $(\beta^i : i < \omega_1)$ of different elements of Γ and vectors $(y^i : i < \omega_1)$ with $y^i \in Y$, $\|y^i\| = 1$ and $y^i_{\beta^i} \neq 0$. By passing to a cofinal subsequence, we can assume that there exists $\varepsilon \in (0, 1)$ such that $|y^i_{\beta^i}| > \varepsilon$ for every $i < \omega_1$. This ε will be fixed from now on in the proof. We can also suppose that the sequence (β^i) is increasing. Set $\varepsilon_m = 2^{-m} \varepsilon^2 / 8$. Recursively on m, we shall define a sequence $S_1 \supset S_2 \supset \cdots S_m \supset \cdots$ of uncountable subsets of ω_1 and a convergent sequence (w_m) of elements of V such that for every m, $P(Y)$ is ε_m-dense in $B_V(w_m, \varepsilon^2)$. After this, $P(Y)$ will be dense in $B_V(\lim w_n, \varepsilon^2/2)$ and the proof will be concluded.

In each inductive step m we shall produce the uncountable set $S_m \subset \omega_1$, and also for every $i \in S_m$ a finite set $a_i^{(m)} \subset \mathrm{supp}(y^i)$ satisfying certain properties. We suppose that we carried out the construction up to the step $m - 1$ and we exhibit how to make step m.

The product of complex numbers is uniformly continuous on bounded sets, hence we can find a number $\eta_m > 0$ such that whenever $\xi, \xi', \zeta, \zeta' \in B_{\mathbb{C}}(0, 1)$ are such that $|\xi - \zeta| < \eta_m$ and $|\xi' - \zeta'| < \eta_m$, then $|\xi\xi' - \zeta\zeta'| < \varepsilon_m/7$. Since $\{\zeta \in \mathbb{C} : \varepsilon < |\zeta| \leq 1\}$ can be covered by finitely many balls of radius η_m, we can find an uncountable subsequence $S' \subset S_{m-1}$ and a complex number ζ_m, with $|\zeta_m| > \varepsilon$, such that $y^i_{\beta_i} \in B(\zeta_m, \eta_m)$ for every $i \in S'$ (hence, whenever $\xi, \xi' \in B(\zeta_m, \eta_m)$ then $|\xi\xi' - \zeta_m^2| < \varepsilon_m/7$).

We take $\delta_m < \varepsilon_m/42$. We know by Proposition 6 that $\|P(x) - P(y)\| < \varepsilon_m/7$ whenever $\|x - y\| < \delta_m$ and $\|x\|, \|y\| \leq 3$, indeed for every homogeneous polynomial P of degree 2 and norm 1.

For every i we fix a finite set $a_i \subset \mathrm{supp}(y^i)$ such that $\beta^i \in a_i$ and

$$\sum_{u \notin a_i} |y^i_u| < \delta_m/2.$$

In other words, if we call x^i to the vector which coincides with y^i on the coordinates of a_i and is supported on a_i, we have that $\|x^i - y^i\| < \delta_m/2$. By virtue of the Δ_m-system lemma, passing to a further cofinal subsequence $S'' \subset S'$ we

can suppose that the sets $\{a_i : i < \omega_1\}$ form a Δ-system with root a. We call $b_i = a_i \setminus a$. We shall suppose that all b_i's have the same cardinality n, and also that $a < b_i < b_j$ whenever $i < j$. Since these finite sets are those defined in step m we may also call $a_i = a_i^{(m)}$, $a = a^{(m)}$, $b_i = b_i^{(m)}$. If $m > 1$, we shall require some coherence properties of the $a_i^{(m)}$'s with respect to the $a_i^{(m-1)}$'s. First, from the very beginning we take $a_i^{(m)} \supset a_i^{(m-1)}$. Also, by passing to a further cofinal subsequence, we suppose that the relative position of $b_i^{(m-1)}$ inside $b_i^{(m)}$ is the same for every i, namely, that if $b_i = \{b_i(0) \prec \ldots \prec b_i(n-1)\}$ then there exists $X \subset n$ such that $b_i^{(m-1)} = \{b_i(r) : r \in X\}$ for every $i \in S''$.

Again, by passing to a further uncountable subsequence, we can suppose that $S'' \subset S_{m-1}$ has the following extra properties:

- There exists $k \in n$ such $\beta^i = b_i(k)$ for every $i \in S''$ (indeed, unless we are in the first step, this is already guaranteed).

- There exists a function $e : a \times n \longrightarrow 2$ such that $f\{\alpha, b_i(r)\} = e(\alpha, r)$ for every $\alpha \in a$, $r \in n$, $i \in S''$.

- There exists a function $d : [n]^2 \longrightarrow 2$ such that $f\{b_i(r), b_i(s)\} = d\{r, s\}$ for every $\{r, s\} \in [n]^2$ and $i \in S''$.

- There exists a function $h : n \times n \setminus \Delta_n \longrightarrow 2$ such that $f(b_i(r), b_j(s)) = h(r, s)$ whenever $i, j \in S_m$, $i < j$ and $r \neq s$. We can do this since f is a partition of the second kind.

We consider the continuous function $\phi : \mathbb{C}^{a \cup n} \times \mathbb{C}^{a \cup n} \longrightarrow V$ given as follows:

$$\phi(z, z') = \sum\{(z_\alpha + z'_\alpha)(z_\beta + z'_\beta)v_{f(\alpha,\beta)} : \{\alpha, \beta\} \in [a]^2\} +$$
$$\sum\{(z_\alpha + z'_\alpha)(z_r + z'_r)v_{e(\alpha,r)} : \alpha \in a, r \in n\} +$$
$$\sum\{(z_r z_s + z'_r z'_s)v_{d(r,s)} : \{r, s\} \in [n]^2\}.$$

That is, ϕ makes what would be the evaluation of the polynomial P on the sum of two vectors supported on two items of the Δ-system, except that it does not compute the summands corresponding to couples $\{b_i(r), b_j(r)\}$ which are the only ones which are possibly different for different couples $i < j$.

Let $\mathcal{B}$ be a countable base for the topology of the space $\mathbb{C}^{a \cup n}$. For every i we can choose $U_i \in \mathcal{B}$ to be a neighborhood of the vector

$$y_*^i = (y_{a(0)}^i, \ldots, y_{a(|a|-1)}^i, y_{b_i(0)}^i, \ldots, y_{b_i(n-1)}^i)$$

such that $|\phi(z) - \phi(w)| < \varepsilon_m/7$ for every $z, w \in U_i \times U_i$. Since $\mathcal{B}$ is countable, we can find an uncountable $S_m \subset S''$ such that:

- There exists $U \in \mathcal{B}$ such that $U_i = U$ for every $i \in S_m$.

Now, we use that f is a partition of the second kind to find ordinals $\{i_p < j_p : p < \omega\} \subset S_m$ such that $f\{b_{i_p}(k), b_{j_p}(k)\} = p$, and in all other cases, we have that $f\{b_{i_p}(r), b_{j_p}(r)\} = 0$.

By the definition of δ_m, we have that $\|P(x^{i_p} + x^{j_p}) - P(y^{i_p} + y^{j_p})\| < \varepsilon_m/7$ since $\|x^i - y^i\| < \delta_m/2$ for every i.

We can compute

$$P(x^{i_p} + x^{j_p}) \quad = \quad \phi(y_*^{i_p}, y_*^{j_p}) + y_{b_{i_p}(k)}^{i_p} y_{b_{j_p}(k)}^{j_p} v_p.$$

On the one hand, $y_{b_i(k)}^i y_{b_j(k)}^j = y_{\beta^i}^i y_{\beta^j}^j \in B(\zeta_m, \eta_m) \cdot B(\zeta_m, \eta_m)$, so

$$|y_{b_i(k)}^i y_{b_j(k)}^j - \zeta_m^2| < \varepsilon_m/7,$$

by the definition of η_m.

On the other hand, since $y_*^{i_p}, y_*^{j_p}, \in U$, we have that

$$|\phi(y_*^{i_p}, y_*^{j_p}) - \phi(y_*^{i_0}, y_*^{j_0})| < \varepsilon_m/7.$$

Set $w_m = \phi(y_*^{i_0}, y_*^{j_0})$. Using that the vectors $\{v_p : p < \omega\}$ are dense in the ball of V, all this data together indicate that the vectors $\{P(y^{i_p} + y^{j_p}) : p < \omega\}$ are $(\varepsilon_m/7 + \varepsilon_m/7 + \varepsilon_m/7)$ - dense in $B_V(w_m, \zeta_m^2) \supset B_V(w_m, \varepsilon^2)$.

We still have to check one fact in order to guarantee that the sequence of the w_m's will be convergent, namely that $\|w_m - w_{m-1}\| < \varepsilon_{m-1} < 2^{-m+1}$. Notice that $w_m = P'(x^{i_0} + x^{j_0})$ where P' is the polynomial obtained from P by deleting the summands corresponding to pairs of ω_1 of the form $\{b_i(r), b_j(r)\}$ for every $i < j$ and every $r < n$. Let us call $x_{(m-1)}^i$, $x_{(m-1)}^j$, $i_0(m-1)$, $j_0(m-1)$, etc. the corresponding objects of the previous step. Notice that $\|x^{i_0} - x_{(m-1)}^{i_0}\| < \delta_{m-1}/2$ and $\|x^{j_0} - x_{(m-1)}^{j_0}\| < \delta_{m-1}/2$ so

$$\|P'(x^{i_0} + x^{j_0}) - P'(x_{(m-1)}^{i_0} + x_{(m-1)}^{j_0})\| < \varepsilon_{m-1}/7.$$

On the other hand, we have that the norm of the difference

$$\|P'(x_{(m-1)}^{i_0} + x_{(m-1)}^{j_0}) - P'(x_{(m-1)}^{i_0(m-1)} + x_{(m-1)}^{j_0(m-1)})\|$$

is equal to

$$\|\phi_{(m-1)}(y_*^{i_0}, y_*^{j_0}) - \phi_{(m-1)}(y_*^{i_0(m-1)}, y_*^{j_0(m-1)})\| < \varepsilon_{m-1}/7,$$

since all those vectors belong to $U_{(m-1)}$. Now, observe that we also have the equality $P'(x_{(m-1)}^{i_0(m-1)} + x_{(m-1)}^{j_0(m-1)}) = w_{m-1}$ thanks to the coherence properties of the sets $a_i^{(m)}$ with respect to the $a_i^{(m-1)}$'s. This finishes the proof. $\square$

6. Results about the existence of large zero subspaces

Let us recall that the sequence $\beth_n$ of cardinals is defined recursively by,

$$\beth_0 = \aleph_0 \text{ and } \beth_{n+1} = 2^{\beth_n}.$$

Lemma 18. *Let n, m be natural numbers and X a Banach space with the property that $dens(X^*, w^*) > \beth_{n+m-1}$. Then every family of $\leq \beth_n$ many homogeneous polynomials of degree $\leq m$ have a common zero subspace of density $\beth_n^+$ (indeed $\beth_{n+1}^+$ for $m \geq 2$).*

Proof: We proceed by induction on m, the case $m = 1$ being clear. Let $\{A_\xi : \xi < \beth_n\}$ be a family of symmetric $\leq m$-linear mappings on X. We construct recursively a sequence $\{x_\alpha : \alpha < \beth_{n+1}^+\}$ of vectors of X as follows: For every $\alpha < \beth_{n+1}^+$, the multilinear maps of the form $A_\xi(x_{\alpha_1}, \ldots, x_{\alpha_k}, *, \ldots, *)$, for $\{\alpha_1, \ldots, \alpha_k\}$ a nonempty family of less than m ordinals less than α and $\xi < \beth_n$, constitute a family of $\leq \beth_{n+1}$ many ($\leq m - 1$)-linear forms. By the inductive hypothesis there is a common zero subspace for all of them of density $\beth_{n+1}^+$, and in particular we can choose x_α in such zero space with $x_\alpha \notin \overline{span}\{x_\beta : \beta < \alpha\}$. The x_α's are almost null for the A_ξ's, the only evaluations which may not vanish are the $A_\xi(x_\alpha, \ldots, x_\alpha) \in \mathbb{C}$. But since α runs up to $\beth_{n+1}^+$ and ξ runs only up to $\beth_n$, we can pass to a subsequence of cardinality $\beth_{n+1}^+$ of the α's where $A_\xi(x_\alpha, \ldots, x_\alpha) = r_\xi$ does not depend on α. Let ζ be a complex number with $\zeta^m = -1$. The span of the vectors $\{x_\alpha + \zeta x_{\alpha+1}\}$ for α even is a common zero subspace for all the multilinear forms A_ξ. $\square$

Corollary 19. *Let X be a Banach space such that $dens(X^*, w^*) > \beth_{m-2}$ and P a homogenous polynomial of degree m on X, then P has a nonseparable zero subspace.*

Proof: Let A be the symmetric multilinear form corresponding to P. We construct inductively vectors $(x_\alpha : \alpha < \omega_1)$ in the following way: For every $\alpha < \omega_1$ we consider all polynomials of the form $A(x_{\alpha_1}, \ldots, x_{\alpha_k}, *, \ldots, *)$ with $\alpha_1, \ldots, \alpha_k < \alpha$. This is a family of $\beth_0$ many homogeneous polynomials of degree $\leq m - 1$, so by the previous lemma they have a common zero subspace of density $\beth_1^+$. In particular we can choose $x_\alpha \notin \overline{span}(x_\beta : \beta < \alpha)$ in this common zero subspace. Again all evaluations of A in the vectors x_α are null except perhaps $A(x_\alpha, \ldots, x_\alpha)$. After multiplication by suitable scalars, we may suppose that this value $A(x_\alpha, \ldots, x_\alpha)$ is constant independent of α. Let ζ be a complex number with $\zeta^m = -1$. The zero subspace for A is then the closed linear span of the vectors $\{x_\alpha + \zeta x_{\alpha+1}\}$ for α even. $\square$

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